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$\mathcal{QT}$-symmetry and weak pseudo-hermiticity

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Abstract

For an invertible (bounded) linear operator $\mathcal{Q}$ acting in a Hilbert space $\mathcal{H}$, we consider the consequences of the $\mathcal{QT}$-symmetry of a non-hermitian Hamiltonian $H : \mathcal{H} \to \mathcal{H}$ where $\mathcal{T}$ is the time-reversal operator. If H is symmetric in the sense that $\mathcal{T}H^{\dagger}\mathcal{T} = H$, then $\mathcal{QT}$-symmetry is equivalent to $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity. But in general this equivalence does not hold. We show this using some specific examples. Among these is a large class of non-$\mathcal{PT}$-symmetric Hamiltonians that share the spectral properties of $\mathcal{PT}$-symmetric Hamiltonians.

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1. Introduction

Among the motivations for the study of the $\mathcal{PT}$-symmetric quantum mechanics is the argument that *$\mathcal{PT}$-symmetry is a more physical condition than hermiticity because $\mathcal{PT}$-symmetry refers to 'spacetime reflection symmetry' whereas hermiticity is 'a mathematical condition whose physical basis is somewhat remote and obscure'* [1]. This statement is based on the assumption that the operators $\mathcal{P}$ and $\mathcal{T}$ continue to keep their standard meanings, as parity (space)-reflection and time-reversal operators, also in $\mathcal{PT}$-symmetric quantum mechanics. But this assumption is not generally true, for unlike $\mathcal{T}$ the parity operator $\mathcal{P}$ loses its connection to physical space once one endows the Hilbert space with an appropriate inner product to reinstate unitarity. This is because for a general $\mathcal{PT}$-symmetric Hamiltonian, such as $H = p^2 + x^2 + ix^3$, the x-operator is no longer a physical observable, the kets $|x\rangle$ do not correspond to localized states in space, and $\mathcal{P} := \int_{-\infty}^{\infty} dx |x\rangle\langle -x|$ does not mean space-reflection [2, 3][1]. Furthermore, it turns out that one cannot actually avoid using the mathematical operations such as hermitian

[1] The space reflection operator is given by $\int_{-\infty}^{\infty} dx\, |\xi^{(x)}\rangle\langle\xi^{(-x)}|$, where $|\xi^{(x)}\rangle$ denote the (localized) eigenkets of the pseudo-hermitian position operator X [2].

conjugation ($A \to A^\dagger$) [2] or transposition ($A \to A^t := \mathcal{T}A^\dagger\mathcal{T}$) in defining the notion of an observable in $\mathcal{PT}$-symmetric quantum mechanics [4, 5].

What makes $\mathcal{PT}$-symmetry interesting is not its physical appeal but the fact that $\mathcal{PT}$ is an antilinear operator[3]. In fact, the spectral properties of $\mathcal{PT}$-symmetric Hamiltonians [6] that have made them a focus of recent interest follow from this property. In general, if a linear operator H commutes with an antilinear operator Θ, the spectrum of H may be shown to be pseudo-real, i.e., as a subset of complex plane it is symmetric about the real axis. In particular, nonreal eigenvalues of H come in complex-conjugate pairs. If H is a diagonalizable operator with a discrete spectrum the latter condition is necessary and sufficient for the pseudo-hermiticity of H [7].

In [8], we showed that the spectrum of the Hamiltonian $H = p^2 + z\delta(x)$ is real and that one can apply the methods of pseudo-hermitian quantum mechanics [2] to identify H with the Hamiltonian of a unitary quantum system provided that the real part of z does not vanish[4]. This Hamiltonian is manifestly non-$\mathcal{PT}$-symmetric. The purpose of this paper is to offer other classes of non-$\mathcal{PT}$-symmetric Hamiltonians that enjoy the same spectral properties.

In the following, we shall use $\mathcal{H}$ and $\mathcal{T}$ to denote a (separable) Hilbert space and an invertible antilinear operator acting in $\mathcal{H}$, respectively. For $\mathcal{H} = L^2(\mathbb{R}^d)$, we define $\mathcal{T}$ by [9]

$$(\mathcal{T}\psi)(\vec{x}) := \psi(\vec{x})^*, \tag{1}$$

for all $\psi \in L^2(\mathbb{R}^d)$ and $\vec{x} \in \mathbb{R}^d$. For $\mathcal{H} = \mathbb{C}^N$, we identify it with complex conjugation: for all $\vec{z} \in \mathbb{C}^N$,

$$\mathcal{T}\vec{z} := \vec{z}^*. \tag{2}$$

2. $\mathcal{QT}$-symmetry

Consider a Hamiltonian operator H acting in $\mathcal{H}$ and commuting with an arbitrary invertible antilinear operator Θ. Because $\mathcal{T}$ is also invertible and antilinear, we can express Θ as $\Theta = \mathcal{QT}$ where $\mathcal{Q} := \Theta\mathcal{T}$ is an invertible linear operator. This suggests the investigation of $\mathcal{QT}$-symmetric Hamiltonians H,

$$[H, \mathcal{QT}] = 0, \tag{3}$$

where $\mathcal{Q}$ is an invertible linear operator. Note that $\mathcal{Q}$ need not be a hermitian operator or an involution, i.e., in general $\mathcal{Q}^\dagger \neq \mathcal{Q}$ and $\mathcal{Q}^2 \neq 1$.

We can easily rewrite (3) in the form

$$\mathcal{T}H\mathcal{T} = \mathcal{Q}^{-1}H\mathcal{Q}. \tag{4}$$

This is similar to the condition that H is $\mathcal{Q}^{-1}$-weakly-pseudo-hermitian [12–15]:

$$H^\dagger = \mathcal{Q}^{-1}H\mathcal{Q}. \tag{5}$$

In fact, (4) and (5) coincide if and only if

$$\mathcal{T}H^\dagger\mathcal{T} = H. \tag{6}$$

The left-hand side of this relation is the usual 'transpose' of H that we denote by H^t. Therefore, $\mathcal{QT}$-symmetry is equivalent to $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity if and only if $H^t = H$, i.e., H

[2] The adjoint $A^\dagger$ of an operator $A : \mathcal{H} \to \mathcal{H}$ is defined by the condition $\langle\psi|A\phi\rangle = \langle A^\dagger|\phi\rangle$, where $\langle\cdot|\cdot\rangle$ is the defining inner product of the Hilbert space $\mathcal{H}$.

[3] This means that $\mathcal{PT}(a_1\psi_1 + a_2\psi_2) = a_1^*\mathcal{PT}\psi_1 + a_2^*\mathcal{PT}\psi_2$, where a_1, a_2 are complex numbers and ψ_1, ψ_2 are state vectors.

[4] Otherwise, H has a spectral singularity and it cannot define a unitary time-evolution regardless of the choice of the inner product.

is symmetric[5]. For example, let $\vec{a}$ and v be respectively complex vector and scalar potentials, $\vec{x} \in \mathbb{R}^d$ and $d \in \mathbb{Z}^+$. Then the Hamiltonian[6]

$$H = \frac{[\vec{p} - \vec{a}(\vec{x})]^2}{2m} + v(\vec{x}) \tag{7}$$

is symmetric if and only if $\vec{a} = \vec{0}$. Supposing that $\vec{a}$ and v are analytic functions, the $\mathcal{QT}$-symmetry of (7), i.e., (4) is equivalent to

$$\frac{[\vec{p} + \vec{a}(\vec{x})^*]^2}{2m} + v(\vec{x})^* = \frac{[\vec{p}_\mathcal{Q} - \vec{a}(\vec{x}_\mathcal{Q})]^2}{2m} + v(\vec{x}_\mathcal{Q}), \tag{8}$$

where for any linear operator $L : \mathcal{H} \to \mathcal{H}$, we have $L_\mathcal{Q} := \mathcal{Q}^{-1} L \mathcal{Q}$. Similarly, the $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity of H, i.e., (5) means

$$\frac{[\vec{p} - \vec{a}(\vec{x})^*]^2}{2m} + v(\vec{x})^* = \frac{[\vec{p}_\mathcal{Q} - \vec{a}(\vec{x}_\mathcal{Q})]^2}{2m} + v(\vec{x}_\mathcal{Q}). \tag{9}$$

As seen from (8) and (9), there is a one-to-one correspondence between $\mathcal{QT}$-symmetric and $\mathcal{Q}^{-1}$-weak-pseudo-hermitian Hamiltonians of the standard form (7), namely that given such a $\mathcal{QT}$-symmetric Hamiltonian H with vector and scalar potentials v and a, there is a $\mathcal{Q}^{-1}$-weak-pseudo-hermitian Hamiltonian H' with vector and scalar potentials $v' = v$ and $a' = \mathrm{i}a$. Note, however, that H and H' are not generally isospectral.

3. A class of matrix models

Consider two-level matrix models defined on the Hilbert space $\mathcal{H} = \mathbb{C}^2$ endowed with the Euclidean inner product $\langle\cdot|\cdot\rangle$. In the following, we explore the $\mathcal{QT}$-symmetry and $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity of a general Hamiltonian $H = \left(\begin{smallmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{smallmatrix}\right)$ for $\mathcal{Q} = \left(\begin{smallmatrix} 1 & 0 \\ \mathfrak{q} & 1 \end{smallmatrix}\right)$, where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{q} \in \mathbb{C}$.

3.1. $\mathcal{QT}$-symmetric two-level systems

Imposing the condition that H is $\mathcal{QT}$-symmetric (i.e., equation (4) holds) restricts $\mathfrak{q}$ to real and imaginary values, and leads to the following forms for the Hamiltonian.

- For real $\mathfrak{q}$:

$$H = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}, \qquad a, c \in \mathbb{R}. \tag{10}$$

  In this case, H is a non-diagonalizable operator with a real spectrum consisting of a.
- For imaginary $\mathfrak{q}$ ($\mathfrak{q} = \mathrm{i}q$ with $q \in \mathbb{R} - \{0\}$):

$$H = \begin{pmatrix} a - \frac{\mathrm{i}}{2}bq & b \\ c + \frac{\mathrm{i}}{2}(a-d)q & d + \frac{\mathrm{i}}{2}bq \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}. \tag{11}$$

  In this case the eigenvalues of H are given by $E_\pm = \frac{1}{2}\left[a + d \pm \sqrt{(a-d)^2 - b(bq^2 - 4c)}\right]$. Therefore, for $(a-d)^2 \geqslant b(bq^2 - 4c)$, H is a diagonalizable operator with a real spectrum;

[5] It is a common practice to identify operators with matrices and define the transpose of an operator H as the operator whose matrix representation is the transpose of the matrix representation of H. Because one must use a basis to determine the matrix representation, unlike $H^t := \mathcal{T} H^\dagger \mathcal{T}$, this definition of transpose is basis-dependent. Note however that H^t agrees with this definition if one uses the position basis $\{|\vec{x}\rangle\}$ in $L^2(\mathbb{R}^d)$ and the standard basis in $\mathbb{C}^N$.

[6] Non-hermitian Hamiltonians of this form have been used in modeling localization effects in condensed matter physics [16].

and for $(a-d)^2 < b(bq^2-4c)$, H is diagonalizable but its spectrum consists of a pair of (complex-conjugate) non-real eigenvalues. Furthermore, the degeneracy condition $(a-d)^2 = b(bq^2-4c)$ marks an exceptional spectral point [10, 11] where H becomes non-diagonalizable. In fact, for $a=d$ and $b=0$ this condition is satisfied and H takes the form (10). Therefore, (11) gives the general form of $\mathcal{QT}$-symmetric Hamiltonians provided that $q\in\mathbb{R}$.

3.2. $\mathcal{Q}^{-1}$-weakly-pseudo-hermitian two-level systems

Demanding that H is $\mathcal{Q}^{-1}$-weakly-pseudo-hermitian does not pose any restriction on the value of $\mathfrak{q}$. It yields the following forms for the Hamiltonian.

- For $\mathfrak{q}=0$:

$$H = \begin{pmatrix} a & b_1+\mathrm{i}b_2 \\ b_1-\mathrm{i}b_2 & d \end{pmatrix}, \qquad a, b_1, b_2, d\in\mathbb{R}. \tag{12}$$

  In this case $\mathcal{Q}$ is the identity operator and $H=H^\dagger$. Therefore, H is a diagonalizable operator with a real spectrum.

- For $\mathfrak{q}\neq 0$:

$$H = \begin{pmatrix} a_1+\mathrm{i}a_2 & -\frac{2\mathrm{i}a_2}{\mathfrak{q}} \\ \frac{2\mathrm{i}a_2}{\mathfrak{q}^*} & a_1-\mathrm{i}a_2 \end{pmatrix}, \qquad a_1, a_2\in\mathbb{R}, \quad \mathfrak{q}\in\mathbb{C}-\{0\}. \tag{13}$$

  In this case the eigenvalues of H are given by $E_\pm = a_1 \pm |a_2||\mathfrak{q}|^{-1}\sqrt{4-|\mathfrak{q}|^2}$. Therefore, for $|\mathfrak{q}|<2$, H is a diagonalizable operator with a real spectrum; and for $|\mathfrak{q}|>2$, H is diagonalizable but its spectrum consists of a pair of (complex-conjugate) non-real eigenvalues. Again the degenerate case: $|\mathfrak{q}|=2$ corresponds to an exceptional point where H becomes non-diagonalizable.

Comparing (11) with (12) and (13) we see that $\mathcal{QT}$-symmetry and $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity are totally different conditions on a general non-symmetric Hamiltonian[7]. For a symmetric Hamiltonian, we can easily show using (12) and (13) that $\mathfrak{q}$ is either real or imaginary and that H takes the form (11). The converse is also true, i.e., any symmetric Hamiltonian of the form (11) is either real (and hence hermitian) or has the form (13). In summary, $\mathcal{QT}$-symmetry and $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity coincide if and only if the Hamiltonian is a symmetric matrix.

4. Unitary $\mathcal{Q}$ and a class of non-$\mathcal{PT}$-symmetric Hamiltonians with a pseudo-real spectrum

If $\mathcal{Q}$ is a unitary operator, the $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity (5) of a Hamiltonian H implies its $\mathcal{Q}$-weak-pseudo-hermiticity, i.e., $H^\dagger = \mathcal{Q}^{-1}H\mathcal{Q}$. This together with (5) leads in turn to

$$[H, \mathcal{Q}^2] = 0, \tag{14}$$

i.e., $\mathcal{Q}^2$ is a symmetry generator. In the following, we examine some simple unitary choices for $\mathcal{Q}$ and determine the form of the $\mathcal{Q}^{-1}$-weak-pseudo-hermitian and $\mathcal{QT}$-symmetric standard Hamiltonians.

[7] Note that this is not in conflict with the fact that in view of the spectral theorems of [12, 17, 18] both of these conditions imply pseudo-hermiticity of the Hamiltonian albeit with respect to a pseudo-metric operator that differs from $\mathcal{Q}^{-1}$ [15].

Consider a standard non-hermitian Hamiltonian (7) in one dimension and let

$$\mathcal{Q} = \mathrm{e}^{\frac{\mathrm{i}\ell p}{\hbar}} \tag{15}$$

for some $\ell \in \mathbb{R}^+$. Then introducing

$$a_1 := \Re(a), \quad a_2 := \Im(a), \quad v_1 := \Re(v), \quad v_2 := \Im(v),$$

where $\Re$ and $\Im$ stand for the real and imaginary parts of their argument, and using the identities

$$\mathcal{Q}^{-1} p \mathcal{Q} = p, \qquad \mathcal{Q}^{-1} x \mathcal{Q} = x - \ell, \tag{16}$$

we can express the condition of the $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity of H, namely (9), in the form

$$a_1(x-\ell) = a_1(x), \qquad a_2(x-\ell) = -a_2(x), \tag{17}$$

$$v_1(x-\ell) = v_1(x), \qquad v_2(x-\ell) = -v_2(x). \tag{18}$$

This means that the real parts of the vector and scalar potentials are periodic functions with period ℓ while their imaginary parts are antiperiodic with period ℓ. This confirms (14), for H is invariant under the translation, $x \to x + 2\ell$, generated by $\mathcal{Q}^2$. We can express a_1, v_1 and a_2, v_2 in terms of their Fourier series. These have, respectively, the following forms

$$\ell\text{-periodic real parts:} \quad \sum_{n=0}^{\infty} \left[c_{1n} \cos\left(\frac{2n\pi x}{\ell}\right) + d_{1n} \sin\left(\frac{2n\pi x}{\ell}\right) \right], \tag{19}$$

$$\ell\text{-antiperiodic imaginary parts:} \quad \sum_{n=0}^{\infty} \left[c_{2n} \cos\left(\frac{(2n+1)\pi x}{\ell}\right) + d_{2n} \sin\left(\frac{(2n+1)\pi x}{\ell}\right) \right], \tag{20}$$

where c_{kn} and d_{kn} are real constants for all $k \in \{1, 2\}$ and $n \in \{0, 1, 2, \ldots\}$.

Conversely, if the real and imaginary parts of both the vector and scalar potentials have respectively the form (19) and (20), the Hamiltonian is $\mathcal{Q}^{-1}$-weak-pseudo-hermitian. In particular its spectrum is pseudo-real; its complex eigenvalues come in complex-conjugate pairs. These Hamiltonians that are generally non-$\mathcal{PT}$-symmetric acquire $\mathcal{QT}$-symmetry provided that they are symmetric, i.e., $a_1 = a_2 = 0$. A simple example is

$$H = \frac{p^2}{2m} + \lambda_1 \sin(2kx) + \mathrm{i}\lambda_2 \cos(5kx),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and $k := \ell^{-1} \in \mathbb{R}^+$.

Next, we examine the condition of $\mathcal{QT}$-symmetry of H, i.e. (8). In view of (16), this condition is equivalent to (18) and

$$a_1(x-\ell) = -a_1(x), \qquad a_2(x-\ell) = a_2(x), \tag{21}$$

which replaces (17). Therefore, v has the same form as for the case of a $\mathcal{Q}^{-1}$-weak-pseudo-hermitian Hamiltonian but a has ℓ-antiperiodic real and ℓ-periodic imaginary parts. In particular, the Fourier series for the real and imaginary parts of a have respectively the form (20) and (19).

We again see that general $\mathcal{QT}$-symmetric Hamiltonians of the standard form (7) are invariant under the translation $x \to x + 2\ell$. This is indeed to be expected, because in view of $[\mathcal{Q}, \mathcal{T}] = 0$ we can express (4) in the form

$$H = \mathcal{Q}^{-1} \mathcal{T} H \mathcal{T} \mathcal{Q} \tag{22}$$

and use this identity to establish

$$\mathcal{Q}^2 H = \mathcal{QT} H \mathcal{TQ} = \mathcal{QT}(\mathcal{Q}^{-1} \mathcal{T} H \mathcal{TQ}) \mathcal{TQ} = H \mathcal{Q}^2.$$

The results obtained in this section admit a direct generalization to higher-dimensional standard Hamiltonians. This involves identifying $\mathcal{Q}$ with a translation operator of the form $e^{\frac{i\vec{\ell}\cdot\vec{p}}{\hbar}}$ for some $\vec{\ell} \in \mathbb{R}^3 - \{\vec{0}\}$. It yields $\mathcal{QT}$-symmetric and $\mathcal{Q}^{-1}$-weakly-pseudo-hermitian Hamiltonians with a pseudo-real spectrum that are invariant under the translation $\vec{x} \to \vec{x} - 2\vec{\ell}$.

An alternative generalization of the results of this section to (two and) three dimensions is to identify $\mathcal{Q}$ with a rotation operator:

$$\mathcal{Q} = e^{\frac{i\varphi\hat{n}\cdot\vec{J}}{\hbar}}, \tag{23}$$

where $\varphi \in (0, 2\pi)$, $\hat{n}$ is a unit vector in $\mathbb{R}^3$ and $\vec{J}$ is the angular momentum operator. Again $[\mathcal{Q}, \mathcal{T}] = 0$ and we obtain generally non-$\mathcal{PT}$-symmetric, $\mathcal{Q}^{-1}$-weak-pseudo-hermitian and $\mathcal{QT}$-symmetric Hamiltonians with a pseudo-real spectrum that are invariant under rotations by an angle 2φ about the axis defined by $\hat{n}$.

Choosing a cylindrical coordinate system whose z-axis is along $\hat{n}$, we can obtain the general form of such standard Hamiltonians.

The $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity of H implies that the real and imaginary parts of the vector and scalar potentials (that we identify with labels 1 and 2, respectively) satisfy

$$\vec{a}_1(\rho, \theta - \varphi, z) = \vec{a}_1(\rho, \theta, z), \qquad \vec{a}_2(\rho, \theta - \varphi, z) = -\vec{a}_2(\rho, \theta, z), \tag{24}$$

$$v_1(\rho, \theta - \varphi, z) = v_1(\rho, \theta, z), \qquad v_2(\rho, \theta - \varphi, z) = -v_1(\rho, \theta, z), \tag{25}$$

where (ρ, θ, z) stand for cylindrical coordinates. Similarly, the $\mathcal{QT}$-symmetry yields (25) and

$$\vec{a}_1(\rho, \theta - \varphi, z) = -\vec{a}_1(\rho, \theta, z), \qquad \vec{a}_2(\rho, \theta - \varphi, z) = \vec{a}_2(\rho, \theta, z). \tag{26}$$

Again we can derive the general form of the Fourier series for these potentials. Here, we suffice to give the form of the general symmetric Hamiltonian:

$$H = \frac{\vec{p}^2}{2m} + \sum_{n=0}^{\infty}[e_n(\rho, z)\cos(2n\omega\theta) + f_n(\rho, z)\sin(2n\omega\theta) \\ + i\{g_n(\rho, z)\cos[(2n+1)\omega\theta] + h_n(\rho, z)\sin[(2n+1)\omega\theta]\}], \tag{27}$$

where e_n, f_n, g_n and h_n are real-valued functions and $\omega := \varphi^{-1} \in \mathbb{R}^+$.

5. A $\mathcal{QT}$-symmetric and non-$\mathcal{PT}$-symmetric Hamiltonian with a real spectrum

In the preceding section we examined $\mathcal{QT}$-symmetric Hamiltonians with a unitary $\mathcal{Q}$. Because $\mathcal{P}$ is also a unitary operator, $\mathcal{QT}$-symmetry with unitary $\mathcal{Q}$ may be considered as a less drastic generalization of $\mathcal{PT}$-symmetry. In this section we explore a $\mathcal{QT}$-symmetric model with a non-unitary $\mathcal{Q}$.

Let a and $a^\dagger$ be the bosonic annihilation and creation operators acting in $L^2(\mathbb{R})$ and satisfying $[a, a^\dagger] = 1$, $\mathfrak{q} \in \mathbb{C} - \{0\}$, and[8]

$$\mathcal{Q} := e^{\mathfrak{q}a}. \tag{28}$$

Consider the Hamiltonian operator

$$H = \alpha a^2 + \beta a^{\dagger 2} + \gamma\{a, a^\dagger\} + \mathfrak{m}a + \mathfrak{n}a^\dagger, \tag{29}$$

where $\alpha, \beta, \gamma, \mathfrak{m}, \mathfrak{n} \in \mathbb{C}$, and demand that H be $\mathcal{QT}$-symmetric. Inserting (29) in (4) and using (1) and the identity $\mathcal{Q}^{-1}a^\dagger\mathcal{Q} = a^\dagger - \mathfrak{q}$, we obtain

$$\alpha^* = \alpha, \qquad \beta^* = \beta, \qquad \gamma^* = \gamma, \qquad \mathfrak{m}^* = \mathfrak{m} - 2\gamma\mathfrak{q}, \qquad \mathfrak{n}^* = \mathfrak{n} - 2\beta\mathfrak{q}, \qquad \mathfrak{n}\mathfrak{q} = 0.$$

[8] The $\mathcal{Q}$ considered in section 3 may be viewed as a fermionic analog of (28).

In particular, because $\mathfrak{q} \neq 0$, we have $\mathfrak{n} = 0$ which in turn implies $\beta = 0$. Furthermore, assuming that $\gamma \neq 0$, we find that $\mathfrak{q}$ must be purely imaginary, $\mathfrak{q} = \mathrm{i}q$ with $q \in \mathbb{R} - \{0\}$, and $\Im(\mathfrak{m}) = \gamma q$. In view of these observations, H takes the following simple form,

$$H = \alpha a^2 + \gamma\{a, a^\dagger\} + (\mu + \mathrm{i}\gamma q)a, \tag{30}$$

where $\mu := \Re(\mathfrak{m})$, $\alpha, \mu \in \mathbb{R}$, and $\gamma, q \in \mathbb{R} - \{0\}$.

Recalling that $\mathcal{P}a\mathcal{P} = -a$ and $\mathcal{T}a\mathcal{T} = a$, we see that for $\mu \neq 0$, the $\mathcal{QT}$-symmetric Hamiltonian (30) is non-$\mathcal{PT}$-symmetric. We also expect that it must have a pseudo-real spectrum. It turns out that actually the spectrum of H can be computed exactly.

To obtain the spectrum of H we use its representation in the basis consisting of the standard normalized eigenvectors $|n\rangle$ of the number operator $a^\dagger a$. Using the following well-known properties of $|n\rangle$ [19],

$$a|n\rangle = \sqrt{n}\,|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle,$$

we find for all $m, n \in \{0, 1, 2, \ldots\}$,

$$H_{mn} := \langle m|H|n\rangle = \gamma(2n+1)\delta_{mn} + (\mu + \mathrm{i}\gamma q)\sqrt{n}\,\delta_{m,n-1} + \alpha\sqrt{n(n-1)}\,\delta_{m,n-2}.$$

As seen from this relation the matrix (H_{mn}) is upper-triangular with distinct diagonal entries and up to three nonzero terms in each row. This implies that the eigenvalues E_n of (H_{mn}) are identical with its diagonal entries, i.e., $E_n = \gamma(2n+1)$. In particular, H has a discrete, equally spaced, real spectrum that is positive for $\gamma > 0$. In the latter case, H is isospectral to a simple harmonic oscillator Hamiltonian with ground-state energy γ.

It is not difficult to see that for each $n \in \{0, 1, 2, \ldots\}$ the span of $\{|0\rangle, |1\rangle, \ldots, |n\rangle\}$, which we denote by $\mathcal{H}_n$, is an invariant subspace of H. This observation allows for the construction of a complete set of eigenvectors of H and establishes the fact that H is a diagonalizable operator with a discrete real spectrum. Therefore, in view of a theorem proven in [20], it is related to a hermitian operator via a similarity transformation, i.e., it is quasi-hermitian [21].

The existence of the invariant subspace $\mathcal{H}_n$ also implies that the eigenvectors $|\psi_n\rangle$ of H corresponding to the eigenvalue E_n belong to $\mathcal{H}_n$, i.e., $|\psi_n\rangle$ is a linear combination of $|0\rangle, |1\rangle, \ldots, |n-1\rangle$ and $|n\rangle$. This in turn allows for a calculation of $|\psi_n\rangle$. For example,

$$|\psi_0\rangle = c_0|0\rangle, \qquad |\psi_1\rangle = c_1\left[|0\rangle + \left(\frac{2\gamma}{\mathfrak{m}}\right)|1\rangle\right],$$

$$|\psi_2\rangle = c_2\left[|0\rangle + \left(\frac{4\gamma\mathfrak{m}}{\mathfrak{m}^2 + \alpha\gamma}\right)|1\rangle + \left(\frac{2\sqrt{2}\gamma^2}{\mathfrak{m}^2 + \alpha\gamma}\right)|2\rangle\right],$$

where c_0, c_1, c_2 are arbitrary nonzero normalization constants and $\mathfrak{m} = \mu + \mathrm{i}\gamma q$.

6. Concluding remarks

It is often stated that $\mathcal{PT}$-symmetry is a special case of pseudo-hermiticity because $\mathcal{PT}$-symmetric Hamiltonians are manifestly $\mathcal{P}$-pseudo-hermitian. This reasoning is only valid for symmetric Hamiltonians H that satisfy $H^\dagger = \mathcal{T}H\mathcal{T}$. In general to establish the claim that $\mathcal{PT}$-symmetry is a special case of pseudo-hermiticity one needs to make use of the spectral theorems of [12, 17, 18]. Indeed what makes $\mathcal{PT}$-symmetric Hamiltonians interesting is the pseudo-reality of their spectrum. This is a general property of all Hamiltonians that are weakly pseudo-hermitian or possess a symmetry that is generated by an invertible antilinear operator. We call the latter $\mathcal{QT}$-symmetric.

In this paper, we have examined in some detail the similarities and differences between $\mathcal{QT}$-symmetry and $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity and obtained large classes of symmetric as well as asymmetric non-$\mathcal{PT}$-symmetric Hamiltonians that share the spectral properties of the $\mathcal{PT}$-symmetric Hamiltonians. In particular, we considered the case that $\mathcal{Q}$ is a unitary operator and showed that in this case $\mathcal{QT}$-symmetry and $\mathcal{Q}^{-1}$-weak-pseudo-hermiticity imply $\mathcal{Q}^2$-symmetry of the Hamiltonian. We also explored a concrete example of a $\mathcal{QT}$-symmetric Hamiltonian with a non-unitary $\mathcal{Q}$ that is not $\mathcal{PT}$-symmetric. We determined the spectrum of this Hamiltonian, established its diagonalizability and showed that it is indeed quasi-hermitian.

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